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On the Noether exponent and other applications of local intersection index

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Abstract

A connection between the index of intersection defined in local analytic geometry and the Noether exponent for germs of holomorphic mappings is established. Also a generalization of estimates for the Łojasiewicz exponent at infinity for polynomial mappings defined on algebraic varieties is commented using local intersection theory methods.

MSC: 32B10; 13H15

Keywords: Intersection index; Analytic cycle; Noether exponent

1. Introduction

Intersection theory in complex analytic geometry unlike algebraic one has rather short history, as its first systematic exposition we find in the paper [7] from 1969. Nevertheless the definition of intersection cycle proposed by R. Draper as taking zero multiplicity for embedded components was quite unsatisfactory. Recently, in 1995 a generalization of this local intersection cycle to the improper case defined through an extended index of intersection (for this introduction shortly: EMI) was introduced in [15]. This construction gives a possibility to look at some important local problems explicitly in the light of analytic geometry and not to go implicitly to apply advanced algebraic machinery. As always in the case of a new theory, the first obvious task is to find its applications. For the moment there are two main directions of this kind of research:

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regular separation of analytic sets and several versions of the Łojasiewicz inequality (see [3,5]) and estimations of the Noether exponent for germs of holomorphic mappings (see [10,4]).

In this paper we proceed in these directions and present two results involving EMI: an estimation for the Noether exponent in the case of an arbitrary (not necessary isolated) improper intersection and a kind of generalization of [5, Theorem 7.3], proved lately also by Jelonek in [10]. As for the first problem it is already known that in the following three special cases: proper intersection [14], improper isolated intersection and so-called quasi-complete intersection [4], the Noether exponent for holomorphic germ is estimated by EMI. As in algebraic case the general situation is much more complicated. Referring to the classic paper [2] we use special properties of the proper projections of analytic sets, a local version of the Chow ideal and finally apply the Briançon–Skoda result and metric criterion for integrality (refreshed recently by Hickel in [8] with some kind of simpler proof) to connect EMI with the Noether exponent. It is worth emphasizing here that – perhaps apart from B–S theorem – our methods are geometrically very simple and intuitive. Some applied properties are generalizations of [3] used in connection with local version of the algebraic translation of [3] given in [11] where global version of Chow ideal appears as conic ideal. In Section 6 we comment how a more careful look at the results from [5] gives a possibility to extend them to the situation of polynomial mappings defined on an algebraic variety.

2. Intersection multiplicity via Tworzewski construction

For the convenience of the reader we outline in this section some basic notions of local intersection theory by Draper and Tworzewski (see [6,7,15] for more details).

2.1. Analytic cycles, their multiplicities and proper intersection

In this paper *analytic* means complex-analytic, and *manifold* means a complex manifold satisfying the second axiom of countability. Let M be a manifold of dimension m. An *analytic cycle* on M is a formal sum

$$A = \sum_{j\in J} \alpha_j Z_j$$

where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{Z_j\}_{j\in J}$ is a locally finite family of distinct irreducible analytic subsets of M.

The analytic set $\sum_{j\in J} Z_j$ is called the *support* of the cycle A and is denoted by $|A|$. If all the components of A have the same dimension k, then A is called a *k-cycle*. We say that A is *positive* if $\alpha_j > 0$ for all $j \in J$.

For a cycle A, we consider the natural extension of the local multiplicity of analytic sets. Namely, if $a \in M$ and $\nu(Z_j, a)$ denotes the multiplicity of Z_j at the point a (see [7]), then the sum

$$\nu(A, a) = \sum_{j\in J} \alpha_j \nu(Z_j, a)$$

is well defined and called the *multiplicity* of A at a.

There exists a unique decomposition

$$A = T_{(m)} + T_{(m-1)} + \cdots + T_{(0)},$$

where $T_{(j)}$ is a j-cycle for $j = 0, \ldots, m$.

Let now X and Y be pure dimensional analytic subsets of M. We say that X and Y *meet properly* on M if $\dim(X \cap Y) = \dim X + \dim Y - m$. Then we have the *intersection product* $X \cdot Y$ of X and Y, which is an analytic cycle on M defined by the formula

$$X \cdot Y = \sum_{Z} i(X \cdot Y, Z)Z,$$

where the summation extends over all analytic components Z of $X \cap Y$ and $i(X \cdot Y, Z)$ denotes the intersection multiplicity along Z in the sense of Draper [7,6]. Such multiplicities are positive integers.

Consider now the special situation when M is a neighbourhood of zero in a normed complex vector space N. Take a pure k-dimensional analytic subset Z of M and a linear subspace L of dimension $m-k$ such that zero is an isolated point of $Z \cap L$. We say that L is a *regular direction* for Z in N if $i(Z \cdot N, 0) = \nu(Z, 0)$. It is known that L is a regular direction for Z if and only if $L \cap C(Z, 0) = \{0\}$, where $C(Z, 0)$ is the tangent cone to Z at zero. Hence the subset of all regular directions for Z in N is open and dense in the grassmannian manifold $G^{m-k}(N)$.

2.2. *Intersections of analytic sets with submanifolds*

Let M be an m-dimensional manifold. Fix a closed s-dimensional submanifold S of M and an open subset U of M such that $U \cap S \neq \emptyset$.

For a given cycle $A = \sum_{j \in J} \alpha_j Z_j$ analytic on M, by its *part supported by* S we mean the cycle $A^S = \sum_{j \in J, Z_j \subset S} \alpha_j Z_j$. Denote by $\mathcal{H}(U)$ the set of all $\mathcal{H} := (H_1, \dots, H_{m-s})$ satisfying the following conditions:

(1) H_j is a smooth hypersurface of U containing $U \cap S$ if $j = 1, \dots, m-s$,
(2) $\bigcap_{j=1}^{m-s} T_x(H_j) = T_x S$ for each $x \in U \cap S$.

For a given analytic subset Z of M of pure dimension k we denote by $\mathcal{H}(U, Z)$ the set of all $\mathcal{H} \in \mathcal{H}(U)$ such that $((U \setminus S) \cap Z) \cap H_1 \cap \dots \cap H_j$ is an analytic subset of $U \setminus S$ of pure dimension $k - j$ (or empty) for $j = 1, \dots, k$. Each system $\mathcal{H} \in \mathcal{H}(U, Z)$ will be called an *admissible system* for the set Z and submanifold S.

Following [15] we present an algorithm which allows us to produce for every $\mathcal{H} \in \mathcal{H}(U, Z)$ an analytic cycle $Z \cdot \mathcal{H}$ in $S \cap U$. In each step of the algorithm we get a cycle $Z_i = Z_i^S + (Z_i - Z_i^S)$. Denote by $i_{\mathcal{H}} \in \{0, \dots, m-s\}$ the first index i for which $|Z_i - Z_i^S| = \emptyset$.

Algorithm 1. *Step 0.* Let $Z_0 = Z \cap U$. Then $Z_0 = Z_0^S + (Z_0 - Z_0^S)$, where Z_0^S is the part of Z_0 supported by $S \cap U$.

Step 1. Let $Z_1 = (Z_0 - Z_0^S) \cdot H_1$. Then $Z_1 = Z_1^S + (Z_1 - Z_1^S)$, where Z_1^S is the part of Z_1 supported by $S \cap U$.

Step 2. Let $Z_2 = (Z_1 - Z_1^S) \cdot H_2$. Then $Z_2 = Z_2^S + (Z_2 - Z_2^S)$, where Z_2^S is the part of Z_2 supported by $S \cap U$.

$\vdots$

Step $i_{\mathcal{H}}$. Let $Z_{i_{\mathcal{H}}} = (Z_{i_{\mathcal{H}}} - Z_{i_{\mathcal{H}}}^S) \cdot H_{i_{\mathcal{H}}}$. Now we have the decomposition

$$Z_{i_{\mathcal{H}}} = Z_{i_{\mathcal{H}}}^S + \left(Z_{i_{\mathcal{H}}} - Z_{i_{\mathcal{H}}}^S\right), \quad \text{and} \quad \left|Z_{i_{\mathcal{H}}} - Z_{i_{\mathcal{H}}}^S\right| \cap S = \emptyset.$$

We call the positive analytic cycle $Z \cdot \mathcal{H} = Z_0^S + Z_1^S + \cdots + Z_{i_{\mathcal{H}}}^S$ in $S \cap U$ the *result* of the above algorithm.

At an arbitrary point $a \in S \cap Z$ we define $g(a)$ as follows. Let

$$\tilde{g}(a) = \tilde{g}(Z, S)(a) := \min_{\text{lex}} \{ \tilde{\nu}(Z \cdot \mathcal{H}, a)\colon \mathcal{H} \in \mathcal{H}(U, Z),\ a \in U \}$$

where $\tilde{\nu}(Z \cdot \mathcal{H}, a) := (\nu(Z_{i_{\mathcal{H}}}^S, a), \ldots, \nu(Z_0^S, a))$ and 'lex' = the lexicographical ordering. Next let $g(a) = g(Z, S)(a) =$ the sum of the coordinates of $\tilde{g}(a)$ – this number is called the *index of intersection* of Z and S at the point a (see [15, Algorithm 4.1]).

Let now M be an open subset of a normed linear space N. We are interested in more convenient systems where hypersurfaces have the form of parts of linear hypersurfaces from the grassmannian manifold $G^{m-1}(N)$.

Theorem 2.1. *(See [1,13].) If S is a linear subspace of a normed linear space M then to obtain the result of the algorithm it suffices to consider the admissible systems $\mathcal{H}$ which consist of parts of linear hypersurfaces. Moreover in such a case the index of intersection of Z and S at a is realized for generic systems of linear hypersurfaces in M.*

Denote by $\mathcal{A}(Z, S)(a) \subset (G^{m-1}(N))^{m-s}$ the set of all admissible linear systems $\mathcal{H}$ for Z and S at the point a such that $\tilde{g}(Z, S)(a) = \tilde{\nu}(Z \cdot \mathcal{H}, a)$.

2.3. General intersection and cycles of zeroes

Let X and Y be irreducible analytic subsets of an m-dimensional manifold M and let $a \in M$. By standard diagonal construction the *multiplicity of intersection* of the sets X and Y at a is defined to be

$$d(a) = d(X, Y)(a) = g\big(X \times Y, \Delta_M, (a, a)\big).$$

The *intersection product* of the irreducible analytic sets X and Y is the unique analytic cycle $X \bullet Y$ in M such that $\nu(X \bullet Y) = d(X, Y)$ (see [15]).

The above definition can be naturally extended to the case of the intersection of a finite number of irreducible analytic subsets and next to arbitrary analytic cycles by multilinearity.

Remark 2.1. (See [1,13].) The index and multiplicity of intersection of an analytic set X with a submanifold S at the point a, and of the analytic set $X \times S$ with the diagonal Δ_M at the point (a, a), coincide.

Roughly speaking we are not obliged to pass by the diagonal construction for the case of intersection with a submanifold. This remark will be very helpful in our considerations of the cycles of zeroes of holomorphic mappings.

Let now U be a neighbourhood of zero in $\mathbb{C}^m$ and $(f_1, \ldots, f_r)\colon U \to \mathbb{C}^r$ be a holomorphic mapping. Consider two analytic subsets of $U \times \mathbb{C}^r$: the graph of f, $G_f = \{(x, y) \in U \times \mathbb{C}^r\colon y_i = f_i(x)\}$ and $Y_f = U \times \{0\}^r$.

Definition 2.1. In this situation we define the cycle Z_f on U given by

$$Z_f \times \{0\} = G_f \bullet Y_f$$

to be the *cycle of zeroes* of f.

Obviously Z_f is an analytic cycle in U but is not necessarily pure dimensional.

3. Special functions for analytic cycle and its Chow ideal

3.1. Distance function and proper projections of analytic cycle

Consider now an analytic k-cycle $A=\sum_{j\in J}\alpha_j Z_j$ in a neighbourhood of zero Ω in $\mathbb{C}^m$ and a linear $(m-k)$-dimensional subspace N of $\mathbb{C}^m$ such that $|A|\cap N=\{0\}$. Then there exists a connected neighbourhood of zero in $\mathbb{C}^m=N^\perp\times N$ of the form $G=U\times W\subset\Omega$ such that the natural projection $\pi_N|_{|A|\cap G}:|A|\cap G\to U$ is a p-sheeted branched covering with $p=\nu(|A|\cdot N,0)$. Without loss of generality we can assume that all the components of A pass through zero and $G=\Omega$ (see [6]). For each component Z_j of A the projection $\pi_N|_{Z_j}$ is also a branched covering and we denote its multiplicity by p_{N,Z_j} or shortly p_j.

For the cycle A we now define a very useful real function on G [3]. First for a component Z_j of the cycle A we put

$$d_{G,N,Z_j}(z)=\prod_{i=1}^{p_j}\bigl|z-z_i^j\bigr| \quad \text{for } (\pi_N|_{Z_j})^{-1}\bigl(\pi_N(z)\bigr)=\bigl\{z_1^j,\dots,z_{p_j}^j\bigr\},$$

where z_i are counted with their multiplicities. Next for the cycle A we take

$$d_{G,N,A}(z)=\prod_j d^{\alpha_j}_{G,N,Z_j}(z).$$

Further we consider the germ of $d_{G,N,A}$ at zero and if convenient take its suitable representatives, denoting them simply by $d_{N,A}$.

In the above situation we take a non-zero linear form $l:N\to\mathbb{C}$ and consider the following mapping:

$$L:N^\perp\times N\ni(x,y)\to\bigl(x,l(y)\bigr)\in N^\perp\times\mathbb{C}.$$

It is easy to see that for each component Z_j of our cycle A $L|_{Z_j}$ is also a $\mu_j=\mu_{N,l,Z_j}$-sheeted analytic covering and $L(Z_j)$ is an irreducible, pure k-dimensional analytic subset in a neighbourhood of zero in a $(k+1)$-dimensional linear space. Additionally also the projection of $L(Z_j)$ on $N^\perp$ is an analytic covering with a multiplicity $s_j:=s_{L(Z_j)}$. This means that there exists a unique system of functions $\alpha_{1,j},\dots,\alpha_{s_j,j}$ holomorphic on U such that in $U\times l(W)$:

$$L(Z_j)=\bigl\{(x,t):\ P_{j,N,l}(x,t)=t^{s_j}+\alpha_{1,j}(x)t^{s_j-1}+\dots+\alpha_{s_j,j}(x)=0\bigr\}.$$

We now introduce another function on G connected with the cycle A (depending also on the subspace N and the form l):

$$F_{N,l,A}(z):=\prod_{j\in J}\bigl(P_{j,N,l}\bigl(L(z)\bigr)\bigr)^{\mu_j\cdot s_j}=\prod_{j\in J}\left(\prod_{i=1}^{s_j}l\bigl(y-y_i^j\bigr)\right)^{\alpha_j},$$

where $(x,y_i^j)=z_i^j$, $i=1,\dots,p_j$.

Combining basic properties of proper projections of analytic sets with some classic linear algebra we can easily obtain the following lemma (see [3, Lemma 3.3] for the proof).

Lemma 3.1. *Let $A=\sum_{j\in J}\alpha_j Z_j$ be a pure k-dimensional analytic cycle on a neighbourhood of zero in $\mathbb{C}^m$, N – $(m-k)$-dimensional linear subspace of $\mathbb{C}^m$ such that $|A|\cap N=\{0\}$, $r=(m-1)\nu(A\cdot N,0)+1$, l, $l_1,\dots,l_r$ – linear forms on N such that $l_{i_1},\dots,l_{i_m}$ are linearly independent for $i_1,\dots,i_m\in\{1,\dots,r\}$.*

Then there exist positive constants c, C such that in an arbitrary small neighbourhood of zero in $\mathbb{C}^m$ the following inequality holds:

$$c\left|F_{N,l,A}(z)\right| \leqslant d_{N,A}(z) \leqslant C \max_i \left|F_{N,l_i,A}(z)\right|.$$

3.2. Chow ideal of analytic cycle

Note that in the previous section we considered a special form of proper projection of analytic cycle on $(k+1)$-dimensional subspace of $\mathbb{C}^m$. Now we will use such projections to regard more closely some algebraic properties of the germ of functions vanishing on the cycle A. For this cycle we call $\pi : \mathbb{C}^m \to \mathbb{C}^{k+1}$ an *admissible projection* for A if 0 is an isolated point of intersection $\operatorname{Ker}\pi \cap |A|$. As above in such a situation there exists a neighbourhood U of zero in $\mathbb{C}^n$ such that $\pi|_{|A|}$ is a proper projection and in particular each $\pi|_{Z_j} : Z_j \to \pi(U)$ is $\mu_{\pi,j}$-sheeted covering. As $\pi(Z_j)$ forms an analytic, irreducible surface in $\pi(U)$ so there exists a unique minimal description of this set, which we denote by $f_{\pi,j}$. Keeping this notation we can state our definition (see also [16] and compare [11] for the global situation).

Definition 3.1. By Chow ideal of the k-cycle A we mean an ideal in $\mathcal{O}_m$ generated by the germs of the following functions

$$F_\pi(z) = \prod_j (f_{\pi,j} \circ \pi)(z)^{\alpha_j \cdot \mu_{\pi,j}}$$

where π goes over all the admissible projections for the cycle A. We denote this ideal by $I_0^{ch}(A)$ or simply $I^{ch}(A)$.

This definition can be naturally extended to the situation of an arbitrary cycle (not necessarily pure dimensional) as: $I^{ch}(A) = \prod_k I^{ch}(A^{(k)})$ where $A^{(k)}$ is the k-dimensional part of the cycle A.

If now $I_0(A)$ denotes an ideal in $\mathcal{O}_m$ of the germs vanishing on the support $|A|$ we have an obvious inclusion $I_0^{ch}(A) \subset I_0(A)$ but the simple example of $A = \{x = y = 0\} + \{x = z = 0\} + \{y = z = 0\}$ [11] shows that it could be no equality.

Proposition 3.1. *Let A be an effective k-cycle in a neighbourhood of zero in $\mathbb{C}^m$, π – an admissible projection for A, L – $(n-k)$-dimensional linear subspace such that $|A| \cap L = \{0\}$, Ω – an open, dense subset of $G^{(m-k)}(\mathbb{C}^m)$.*

Then the following inequalities remain true in an arbitrary small neighbourhood of zero:

(1) *if* $\operatorname{Ker}\pi \subset L$, *then* $\exists c_1 > 0$: $|F_\pi(z)| \leqslant c_1 d_{L,A}(z)$,
(2) *there exist $G_1, \ldots, G_s$ – generators of Chow ideal for A, $c_2 > 0$ such that*

$$d_{L,A}(z) \leqslant c_2 \max_i \left|G_i(z)\right|,$$

(3) *there exist $N_1, \ldots, N_r \in \Omega$, $c_3 > 0$ such that*:

$$d_{L,A}(z) \leqslant c_3 \max_i d_{N_i,A}(z).$$

Proof. As the functions $F_{N,l,A}$ introduced in Section 3.1 are obviously generators of Chow ideal for the cycle A the first two properties are straightforward consequences of Lemma 3.1.

Now we will prove the third one. Let B be the set of all systems $(\varphi_1, \dots, \varphi_k)$ of k linear forms on $\mathbb{C}^m$ such that $\{\varphi_1(z) = \dots = \varphi_k(z) = 0\} \in \Omega$. Without loss of generality we can assume that $L = \{0\} \times \mathbb{C}^{m-k} = \{z \in \mathbb{C}^m : z_1 = \dots = z_k = 0\}$ and take the subspace $\tilde{L} = \{z \in \mathbb{C}^m : z_1 = \dots = z_{k-1} = 0\}$. As the set Ω is open and dense in $G^{m-k}(\mathbb{C}^m)$ we can choose a system of hyperplanes $\zeta_j = \{z \in \mathbb{C}^n : l_j(z) = 0\}$, $j = 1, \dots, r$, where $r = (n-k-1)\nu(A \cdot L, 0) + 1$ satisfying the following conditions:

(1) $\zeta_j \in p_k(B)$, where p_k is the projection on the last coordinate,
(2) every system of linear forms $l_{j_1}|_L, \dots, l_{j_{m-k}}|_L$ is linearly independent for $j_1, \dots, j_{m-k} \in \{1, \dots, r\}$ provided $j_v \neq j_t$ for $v \neq t$.

By Lemma 3.1 applied to the space L and the system $l_1|_L, \dots, l_r|_L$ we get

$$d_{L,A}(z) \leqslant c' \max_j \left| F_{L, l_j|_L, A}(z) \right|$$

in an arbitrary small neighbourhood of zero in $\mathbb{C}^n$, for some constant $c' > 0$.

Consider now the intersection $L_j = \zeta_j \cap \tilde{L}$. For each of the epimorphisms $\Phi_j : \mathbb{C}^m \ni (z_1, \dots, z_m) \to (z_1, \dots, z_k, l_j|_L(z_{k+1}, \dots, z_m)) \in \mathbb{C}^{k+1}$ we have $\ker \Phi_j \subset L_j$. Since $\dim(\ker \Phi_j) = m-k-1$ it is possible to choose for every $j \in \{1, \dots, r\}$ a linear form $\tilde{l}_j$ on L_j such that the equality $\ker \Phi_j = \ker \tilde{\Phi}_j$ for $\tilde{\Phi}_j : \mathbb{C}^m = L_k^{\perp} \times L_j \ni (x, y) \to (x, \tilde{l}_j(y)) \in \mathbb{C}^{k+1}$ holds. Consequently, there exist linear isomorphisms $I_j : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ for which $\tilde{\Phi}_j = I_j \circ \Phi_j$. Hence, for every component Z of the cycle A the multiplicities $\mu_{\Phi_j, Z}$ and $\mu_{\tilde{\Phi}_j, Z}$ coincide. As the germs of $P_{l_j|_L, Z}$ and $P_{\tilde{l}_j, Z} \circ I$ at zero in $\mathbb{C}^{k+1}$ generate the ideal of the germ of $\Phi_j(Z)$, we get:

$$\left| F_{l_j|_L, A}(z) \right| \leqslant \tilde{c} \left| F_{\tilde{l}_j, A}(z) \right| \quad \text{for some } \tilde{c} > 0.$$

Combining now two previous inequalities and applying once more Lemma 3.1 to each of the forms $\tilde{l}_j$ we finally get $d_{L,A}(z) \leqslant c \max_i d_{L_j, A}(z)$ for some constant $c > 0$.

Now instead of L we take each of the $L_j = \{l_j(z) = z_1 = \dots = z_{k-1} = 0\}$ and put $\tilde{L}_j = \{z \in \mathbb{C}^m \colon l_j(z) = z_1 = \dots = z_{k-2} = 0\}$. Again, as the set Ω is open and dense we can choose a system of hyperplanes $\zeta_{j,i} = \{z \in \mathbb{C}^m \colon l_{j,i}(z) = 0\}$, $i = 1, \dots, r_j$, where $r_j = (m-k-1)\nu(A \cdot L_j, 0) + 1$ satisfying the following conditions:

(1) $(\zeta_{j,i}, \zeta_j) \in p_{k,k-1}(B)$, where $p_{k,k-1}$ is the projection on the last two coordinates,
(2) every system of linear forms $l_{j,i_1}|_L, \dots, l_{j,i_{n-k}}|_L$ is linearly independent for $(j, i_1), \dots, (j, i_{n-k}) \in \{1, \dots, r_j\}$ provided $(j, i_v) \neq (j, i_t)$ for $v \neq t$.

Taking $L_{j,t} := \zeta_{j,t} \cap \tilde{L}_j$ we next repeat previous considerations obtaining finally in an arbitrary small neighbourhood of zero an inequality of the following form:

$$d_{L_j, A}(z) \leqslant c_j \max_t d_{L_{j,t}, A}(z).$$

Continuing in the same way we get our result after 'exchanging' all the coordinates. □

4. Integral closure of ideals

We recall now two important criteria of integrality over an ideal: an algebraic and a geometric one. For our purpose especially the well-known metric criterion recently reproved by Hickel in [8] is interesting (we give a simpler version restricted to our situation).

Theorem 4.1. *(See [8].) Let $I = (f_1, \dots, f_n)$ be an ideal in $\mathcal{O}_m$ and $g \in \mathcal{O}_m$. Then the following conditions are equivalent*

(1) $g \in \bar{I}$ *(where $\bar{I}$ denotes the integral closure of I),*
(2) $\exists \varepsilon > 0$, $\exists C > 0$: $\forall |z| < \varepsilon$: $|g(z)| \leqslant C \cdot \max_i |f_i(z)|$,
(3) *for each $\varphi : \mathcal{O}_m \to \mathbb{C}[\![t]\!]$ a local morphism of $\mathbb{C}$-algebras the following inequality holds*

$$\operatorname{ord}_t \big(\varphi(g)\big) \geqslant \min_i \operatorname{ord}_t \big(\varphi(f_i)\big).$$

The classic Briançon–Skoda Theorem fixes a connection between the ideal and its integral closure. We present below its generalization from [12].

Theorem 4.2. *Let R be a local, m-dimensional, regular ring, I – an ideal in R, $k \in \mathbb{N}^\star$. Then $\overline{I^{k+\mu-1}} \subset I^k$ for $\mu = \min(m, s)$, where s is the minimal number of generators of the ideal I.*

That means that for an ideal I of the ring $R = \mathcal{O}_m$ which is an m-dimensional, local, regular ring we have: $(\bar{I})^\mu \subset \overline{I^\mu} \subset I$.

Let now $A = \sum_{j \in J} \alpha_j Z_j$ be a k-cycle in the neighbourhood of zero in $\mathbb{C}^m$. Observe that Proposition 3.1 implies the following property of the ideals connected with it.

Proposition 4.1. $\prod_{j \in J} I(Z_j)^{\alpha_j \nu(Z_j,0)} \subset \overline{I^{ch}(A)}$.

Proof. Let g_j be holomorphic functions in a neighbourhood of zero in $\mathbb{C}^m$ with $g_j|_{|Z_j|} \equiv 0$. It suffices to show that there exist $f_1, \dots, f_l \in I^{ch}(A)$ such that $\prod_{j \in J} |g_j(z)|^{\alpha_j} \leqslant \max_{i=1}^l |f_i(z)|$ near zero.

Let N be an $(m-k)$-dimensional linear subspace of $\mathbb{C}^m$ such that it is a regular direction for each of the components Z_j. If now $(\pi_N|_{Z_j})^{-1}(z) = \{z_1^j, \dots, z_{\nu(Z_j,0)}^j\}$ then we get for a constant $C > 0$ and some generators $f_1, \dots, f_l$ of the ideal $I^{ch}(A)$ the following inequality in a neighbourhood of zero in $\mathbb{C}^m$

$$\begin{aligned}\prod_{j \in J} \big|g_j(z)\big|^{\nu(Z_j,0)\cdot\alpha_j} &= \prod_{j \in J} \big(\big|g(z) - g\big(z_1^j\big)\big| \cdot \dots \cdot \big|g(z) - g\big(z_{\nu(Z_j,0)}^j\big)\big|\big)^{\alpha_j} \\ &\leqslant C d_{N,A}(z) \leqslant \max_{i=1}^{l} \big|f_i(z)\big|\end{aligned}$$

where the last one comes from Proposition 3.1. □

Directly from definition of the Chow ideal for analytic cycle $A = \sum_{j \in J} \alpha_j Z_j$ and the basic properties of multiplicity we have the inclusion $I^{ch}(A) \subset \prod_{j \in J} I^{ch}(Z_j)^{\alpha_j}$ but the same example as given before Proposition 3.1 shows that the equality is not always true. Nevertheless using similar idea applied in [11] to global case we can prove the following property of integral closures.

Theorem 4.3. *For an analytic cycle $A = \sum_{j\in J} \alpha_j Z_j$ in a neighbourhood of zero of $\mathbb{C}^m$ we have $\overline{I^{ch}(A)} = \overline{\prod_{j\in J} I^{ch}(Z_j)^{\alpha_j}}$.*

Proof. First make some general comments. Fix an arbitrary pure k-dimensional analytic cycle X in a neighbourhood of zero in $\mathbb{C}^m$. Let $\varphi : \mathcal{O}_m \to \mathbb{C}[\![t]\!]$ be a local morphism of $\mathbb{C}$-algebras and $F_X : \mathcal{B}^{k+1}(\mathbb{C}^m, X) \ni \pi \to f_{\pi,X} \in \mathcal{O}_m$ where $B^{k+1}(\mathbb{C}^m, X) \subset G^{k+1}(\mathbb{C}^m)$ is the subset of admissible projections for the cycle X and $f_{\pi,X}$ is the corresponding description of the image of X. As the mapping $G_X := \mathrm{ord} \circ \varphi \circ F_X : \mathcal{B}^{k+1}(\mathbb{C}^m, X) \to \mathbb{Z}$ is an upper semicontinuous discrete function it achieves the minimum value on a dense open subset of $\mathcal{B}^{k+1}(\mathbb{C}^m, X)$.

Returning now to our proof notice that by definition of the Chow ideal (see Definition 3.1) it suffices to consider a pure k-dimensional cycle A and to show that if $A = B + C$ where B, C are also pure k-dimensional cycles then $I^{ch}(A) I^{ch}(B) \subset \overline{I^{ch}(A)}$.

Let π_1, π_2 be admissible projections for B and C respectively. Take $\overline{\pi_0}$ such that $G_B(\overline{\pi_0}) = \min\{G_B(\pi),\ \pi \in \mathcal{B}^{k+1}(\mathbb{C}^m, B)\} \leqslant G_B(\pi_1)$ and let U be an open neighbourhood of $\overline{\pi_0}$ in $\mathcal{B}^{k+1}(\mathbb{C}^m, B)$ such that for $\pi \in U$ holds $G_B(\overline{\pi_0}) = G_B(\pi)$. As $\mathcal{B}^{k+1}(\mathbb{C}^m, B)$ is a dense open subset of $G^{k+1}(\mathbb{C}^m)$ there exists $\pi_0 \in U$ such that $G_C(\pi_0) \leqslant G_C(\pi_2)$ and $G_B(\pi_0) \leqslant G_A(\pi_1)$ so finally $G_B(\pi_1) G_C(\pi_2) \geqslant G_B(\pi_0) G_C(\pi_0)$. Now as π_0 is an admissible projection for the cycle $A = B + C$ it suffices to apply the third condition from Theorem 4.1 to end the proof. □

5. Noether exponent

Theorem 5.1. *Let Z be a pure k-dimensional analytic subset of $\mathbb{C}^m$, S a closed s-dimensional submanifold of $\mathbb{C}^m$ and $0 \in Z \cap S$. Let U be a neighbourhood of zero, $\mathcal{H}(U, Z)$ an admissible system for Z and S (see Section 2) such that for every $i \in \{1, \ldots, m-s\}$ all the components of $|Z_i^S|$ pass through zero, g_i – holomorphic mappings in $U \cap S$ such that $g_i|_{|Z_i^S|} \equiv 0$ for $i = 1, \ldots, m-s$.*

Then there exists $(F_1, \ldots, F_t)$ a system of generators of the Chow ideal $I^{ch}(Z)$ such that in a neighbourhood of zero for $z \in Z \cap S$ the following inequality holds:

$$\left| \prod_{i=1}^{r} g_i^{\nu(Z_i^S, 0)}(z) \right| \leqslant \max_j \left| F_j(z) \right|.$$

Proof. Set $n := i_{\mathcal{H}}$ and applying an appropriate chart assume $H_1, \ldots, H_n$ be linear subspaces. Fix a linear subspace L_n in $H_1 \cap \cdots \cap H_n$ which is regular direction for $|Z_n^S|$ in $H_1 \cap \cdots \cap H_n$. Modifying slightly the proof of [3, 4.4] we choose a special system of linear subspaces in $\mathbb{C}^m$ in n steps.

(1) Applying Proposition 3.1(3) to L_n we find a neighbourhood $U_{n-1} \subset U$ of zero in $\mathbb{C}^m$ and regular directions $L_{(n-1,1)}, \ldots, L_{(n-1,s_n)}$ for the set $|Z_{n-1}|$ in $H_1 \cap \cdots \cap H_{n-1}$ such that if $z \in U_{n-1} \cap H_1 \cap \cdots \cap H_n \subset H_1 \cap \cdots \cap H_{n-1}$ (for some representatives of germs of involved distance functions) then

(i) $d_{L_n, Z_{n-1} - Z_{n-1}^S} \leqslant c_{n-1} \max_i d_{L_{(n-1,i)}, Z_{n-1} - Z_{n-1}^S}(z)$,
(ii) $\nu(|Z_{n-1}| \cdot L_{(n-1,i)}, 0) = \nu(|Z_{n-1}|, 0)$.

(2) Applying once more Proposition 3.1(3) to the cycle $Z_{n-2} - Z_{n-2}^S$ and each of the subspaces $L_{(n-1,i)}$ in $H_1 \cap \cdots \cap H_{n-1}$ we find regular directions $L_{(n-2,i,1)}, \ldots, L_{(n-2,i,s_{(n-1,i)})}$ which

are the $(m-k)$-dimensional linear subspaces in $H_1 \cap \cdots \cap H_{n-2}$. We choose them in the way that for a neighbourhood $U_{(n-2,i)} \subset U_{n-1}$ and some $c_{(n-2,i)}$ for $z \in U_{(n-1,i)} \cap H_1 \cap \cdots \cap H_{n-1} \subset H_1 \cap \cdots \cap H_{n-2}$ holds:

(i) $d_{L_{(n-1,i)}, Z_{n-2}-Z^S_{n-2}}(z) \leqslant c_{(n-2,i)} \max_j d_{L_{(n-2,i)}, Z_{n-2}-Z^S_{n-2}}(z)$,
(ii) $\nu(|Z_{n-2}| \cdot L_{(n-1,i,j)}, 0) = \nu(|Z_{n-2}|, 0)$.

Put $U_{(n-2)} = \bigcap_{i=1}^{s_n} U_{(n-2,i)}$, $C_{(n-2)} = \max_i c_{(n-2,i)}$, $x_{n-1} = \max_i s_{(n-1,i)}$.

We continue in the same way and finally in the last nth step we get s_1 linear subspaces $L_{(0,I)}$, where $I = I_n = (i_1, \ldots, i_n)$ which are regular directions for $|Z_0|$ in the space $\mathbb{C}^m$.

Next applying to each of $L_{(0,I)}$ the property (2) of Proposition 3.1 we choose a system $(H_1, \ldots, H_t)$ of generators of the Chow ideal of the cycle Z_0 such that for a constant C (independent of I) and for $z \in W$ a neighbourhood of zero in $\mathbb{C}^m$ (for all I) we get the following inequality:

$$d_{L_{(0,I)}, Z_0}(z) \leqslant C \max_{j=1}^{t} \left| H_j(z) \right|.$$

Fix now $z \in W \cap S$, then we have $d_{L_n, Z_{n-1}-Z^S_{n-1}}(z) = d_{L_n, Z^S_n}(z)$.

(1) From the system $(L_{(n-1,1)}, \ldots, L_{(n-1,s_n)})$ we choose the 'maximal' subspace for the point z denoted by $L_{n-1}(z)$ for which holds:

$$d_{L_n, Z_{n-1}-Z^S_{n-1}}(z) \leqslant C_{n-1} d_{L_{n-1}(z), Z_{n-1}-Z^S_{n-1}}(z)$$

where $C_{n-1} = c_{n-1}s_n$.

As $z \in H_1 \cap \cdots \cap H_{n-1}$ so $d_{L_{n-1}(z), Z_{n-1}-Z^S_{n-1}}(z) \cdot d_{L_{n-1}(z), Z^S_{n-1}}(z) = d_{L_{n-1}(z), Z_{n-2}-Z^S_{n-2}}(z)$ and consequently:

$$d_{L_n, Z^S_n}(z) \cdot d_{L_{n-1}(z), Z^S_{n-1}}(z) \leqslant C_{n-1} d_{L_{n-1}(z), Z_{n-1}-Z^S_{n-1}}(z).$$

Repeating the same considerations in the next steps we finally get the following inequality (for fixed $z \in W \cap S$):

$$\begin{aligned} &d_{L_n, Z^S_n}(z) \cdot d_{L_{n-1}(z), Z^S_{n-1}}(z) \cdot \cdots \cdot d_{L_0(z), Z^S_0}(z) \\ &\quad \leqslant C_{n-1} \cdot \cdots \cdot C_0 d_{L_0(z), Z_0}(z) \leqslant C_{n-1} \cdots C_0 \cdot C \max_j \left| H_j(z) \right|. \end{aligned}$$

As we see all the constants in the last inequality are independent of $z \in W \cap S$. To finish the proof it suffices to notice that all the points of the fibers of projections considered in the function d_{L_i, Z^S_i} lie in the zero set of g_i so applying the mean value theorem we get the result. □

Corollary 5.1. *Let $f = (f_1, \ldots, f_p) : U \to \mathbb{C}^p$ be a holomorphic mapping where U is an open neighbourhood of $0 \in \mathbb{C}^n$. If $g : U \to \mathbb{C}$ is a holomorphic function vanishing on the zero set of $(f_1, \ldots, f_p)$ then $g_0^d \in I_0(f_1, \ldots, f_p)$ for $d = \nu(Z_f, 0) \cdot \min\{p, n\}$.*

Proof. Put $S = U \times \{0\}$, $G_f = \{(x, f(x)) \in U \times \mathbb{C}^p\}$ and let $\mathcal{H} = (H_1, \ldots, H_p) \in \mathcal{A}(G_f, S)(0)$ (cf. Theorem 2.1). First notice that without loss of generality we can assume that in the neighbourhood U we have $H_i = \{y_i = 0\}$. In view of Theorem 4.1(2) it suffices now to apply Theorem 5.1 for $Z = G_f$ and $\mathcal{H}$ to get that $g_0^d \in \overline{I_0(f_1, \ldots, f_p)}$ (as the Chow ideal of the graph is contained in the ideal of the graph and for $z = (x, y_1, \ldots, y_p) \in Z \cap S$ we have $y_i = 0$). Briançon–Skoda result ends the proof. □

6. Remark on the Łojasiewicz exponent at infinity

In the paper [5] an estimation for the Łojasiewicz exponent for polynomial mappings on $\mathbb{C}^n$ was proved by the means of intersection theory and regular separation at infinity of algebraic sets. A generalization of this property to polynomial mappings with finite number of zeros on an affine s-dimensional variety of degree D was lately derived as a consequence of the work [9]. Below we see how the same result can be very easily seen by the same considerations as in [5].

First recall that for two non-empty closed subsets of an m-dimensional normed complex vector space and $q \in (-\infty, 1]$ we say that X and Y are q-separated at infinity if there exist $c, R > 0$ such that we have

$$\varrho(z, X) + \varrho(z, Y) \geqslant c|z|^q$$

provided $|z| > R$. Next cite two key results from [5].

Lemma 6.1. *(See [5, 5.2].) Closed non-empty subsets X and Y of M are q-separated at infinity if and only if there exist $c, R > 0$, such that for $x \in X$, $|x| > R$ we have*

$$\varrho(x, Y) \geqslant c|x|^q.$$

Theorem 6.1. *(See [5, 6.1].) If X and Y are non-empty algebraic subsets of the space M of pure dimensions and $\dim(X \cap Y) = 0$, then X and Y are q-separated at infinity for $q = 1 - (\deg X)(\deg Y) + \deg(X \bullet Y)$.*

Now we can prove the following generalization of [5, 7.3].

Theorem 6.2. *Let $Z \subset \mathbb{C}^m$ be an affine n-dimensional algebraic affine variety of degree D. Assume that $F = (f_1, \dots, f_k): \mathbb{C}^m \to \mathbb{C}^k$ is a polynomial mapping with finitely many zeroes on Z. Let $d_i = \deg f_i > 0$ (where $d_1 \geqslant d_2 \geqslant \dots \geqslant d_k$). Then there exists a constant $C > 0$ such that*

$$\mathcal{L}_\infty(F, Z) \geqslant d_k - D \cdot B(d_1, \dots, d_k, m) + \sum_{b \in F^{-1}(0) \cap Z} \mu_b(F|_Z),$$

where

$$B(d_1, \dots, d_k, m) = \begin{cases} d_1 \cdot \dots \cdot d_{k-1} \cdot d_k, & \text{if } k \leqslant m; \\ d_1 \cdot \dots \cdot d_{m-1} \cdot d_k, & \text{if } k > m \end{cases}$$

and $\mathcal{L}_\infty(F, Z) = \sup\{s \in \mathbb{R}\colon \exists A > 0,\ r > 0\colon (x \in Z,\ |x| > r) \Rightarrow (|F(x)| \geqslant A|x|^s)\}$ is the Łojasiewicz exponent of F on Z at infinity.

Proof. Applying standard reduction we can restrict our attention to the case $k \leqslant m$. First consider a linear form $L: \mathbb{C}^m \to \mathbb{C}$ such that $F^{-1}(0) \cap Z \cap \ker L = \emptyset$, put $d := d_k$ and consider X the algebraic subset of $\mathbb{C}^m \times \mathbb{C}^k$ of pure dimension m defined by the equations: $L^{d-1}(x) y_i = f_i(x), i = 1, \dots, k$. Because of the properties of d_i we obtain that the degree of this set is less or equal to $B(d_1, \dots, d_k, m)$. Additionally from construction the intersection cycle of X and Y is just the zero cycle of $F|_Z$.

By Theorem 6.1 applied to X and $Y = Z \times \{0\} \subset \mathbb{C}^m \times \mathbb{C}^k$ we know that they are q-separated at infinity with

$$q = 1 - D \cdot B(d_1, \dots, d_k, m) + \deg(X \bullet Y) = \sum_{b \in F^{-1}(0) \cap Z} \mu_b(F|_Z).$$

Now from Lemma 6.1 we get $r_L > 1$, $c_L > 0$ such that for $|x| > r_L$ and $x \in Z$ the inequality $|F(x)| \geqslant c_L(|x| + |F(x)|)^q > A_L|x|^q$ where $A_L = c_L$ if $q \geqslant 0$ and $A_L = \min\{2^q c_L, 1\}$ if $q < 0$.

Without loss of generality we assume that $F^{-1}(0) \cap Z \cap \{x_i = 0\} = \emptyset$ for $i = 1, \ldots, m$ so it is possible to repeat the above considerations for the forms $L_i(x) = x_i$ and this way obtain $r > 0$, $K > 0$ such that $|F(x)| \geqslant K|x_j|^{d-1}|x|^q$ for $|x| > r$ and $x \in Z$. Summing up we have $|F(x)| \geqslant A|x|^{d-1+q}$ for $x \in Z \cap \{|x| > r\}$ and this ends the proof. $\square$

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